Topic: Finding the value of an inverse trig function

Question: Find the value of the inverse trig function.

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

Answer choices:

A $-\frac{\pi}{3}$

B $\frac{5\pi}{6}$

C $\frac{2\pi}{3}$

D $-\frac{\pi}{6}$

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Solution: B

The range of the function $\cos^{-1}(x)$ is the interval $[0,\pi]$, so we need to find the angle θ in the interval $[0,\pi]$ for which

$$\cos\theta = -\frac{\sqrt{3}}{2}$$

In fact, θ is in the interval $(\pi/2,\pi)$, since its cosine is negative.

Recall that

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

By the difference identity for cosine,

$$\cos\left(\pi - \frac{\pi}{6}\right) = \cos\pi\cos\frac{\pi}{6} + \sin\pi\sin\frac{\pi}{6}$$

$$\cos\left(\pi - \frac{\pi}{6}\right) = (-1)\cos\frac{\pi}{6} + (0)\sin\frac{\pi}{6}$$

$$\cos\left(\pi - \frac{\pi}{6}\right) = -\cos\frac{\pi}{6}$$

$$\cos\left(\pi - \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

Now

$$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

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so

$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$

Note that $5\pi/6$ is in the interval $(\pi/2,\pi)$, hence that

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

Another way to get the answer to this question, once we recognize that

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

is to apply the reference angle theorem. The reference angle for $\pi/6$ is equal to the reference angle for $\pi - (\pi/6) = 5\pi/6$. Since the angle of measure $5\pi/6$ is in the second quadrant (and hence its cosine is negative), this tells us that

$$\cos\frac{5\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

From this it follows that

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

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Topic: Finding the value of an inverse trig function

Question: Find the value of the inverse trig function.

$$\tan^{-1}\left(-\sqrt{3}\right)$$

Answer choices:

A $-\frac{\pi}{3}$

B $-\frac{\pi}{6}$

C $\frac{2\pi}{3}$

D $\frac{5\pi}{3}$

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Solution: A

The range of the function $\tan^{-1}(x)$ is the interval $(-\pi/2,\pi/2)$, so we need to find the angle θ in the interval $(-\pi/2,\pi/2)$ for which $\tan\theta = -\sqrt{3}$. Since $-\sqrt{3}$ is negative, we need to find the angle θ in the interval $(-\pi/2,0)$ for which $\tan\theta = -\sqrt{3}$.

Note that

$$\sqrt{3} = \frac{\sqrt{3}}{1} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

Since $\sqrt{3}$ is positive, we know that $\cot^{-1}\left(\sqrt{3}\right)$ is in the first quadrant. Thus we'll first find the angle α in the interval $(0,\pi/2)$ for which $\tan(\alpha) = \sqrt{3}$.

Recall that

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

and

$$\cos\frac{\pi}{3} = \frac{1}{2}$$

Thus

$$\tan\frac{\pi}{3} = \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

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From this result and the fact that $\pi/3$ is in the interval $(0,\pi/2)$, we have that $\alpha = \pi/3$.

Since

$$\tan(-\alpha) = \frac{\sin(-\alpha)}{\cos(-\alpha)} = \frac{-\sin\alpha}{\cos\alpha} = -\frac{\sin\alpha}{\cos\alpha} = -\tan\alpha$$

we obtain

$$\tan\left(-\frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}$$

Now $-\pi/3$ is in the interval $(-\pi/2,0)$, so

$$\theta = -\alpha = -\frac{\pi}{3}$$

From this it follows that

$$\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

Note that answer choice D (namely, $5\pi/3$) is the measure of an angle which is coterminal with the angle of measure $-\pi/3$, so

$$\tan\frac{5\pi}{3} = \tan\left(-\frac{\pi}{3}\right)$$

However, $5\pi/3$ isn't in the interval $(-\pi/2,\pi/2)$, so it isn't in the range of the function $\tan^{-1}(x)$.

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Topic: Finding the value of an inverse trig function

Question: Which of the following is the range of an inverse of the function?

$$\sin\left(\theta - \frac{3\pi}{8}\right)$$

Answer choices:

A The interval $\left[-\frac{5\pi}{8}, -\frac{\pi}{4}\right]$

B The union of the intervals $\left[-\pi, -\frac{\pi}{8}\right]$ and $\left[0, \frac{7\pi}{8}\right]$

C The interval $\left[-\frac{\pi}{8}, \frac{7\pi}{8}\right]$

D The union of the intervals $\left[-\pi, -\frac{3\pi}{8}\right]$ and $\left[0, \frac{5\pi}{8}\right]$

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Solution: C

The range of the function $\sin^{-1} x$ is the interval $[-\pi/2,\pi/2]$, that is, for every x in the interval $[-1,1]$, there is a unique angle θ in the interval $[-\pi/2,\pi/2]$ such that $\sin\theta = x$.

Thus for every x in the interval $[-1,1]$, there is a unique angle θ such that $\theta - (3\pi/8)$ is in the interval $[-\pi/2,\pi/2]$, and

$$\sin\left(\theta - \frac{3\pi}{8}\right) = x$$

Equivalently, for every x in the interval $[-1,1]$, there is a unique angle θ such that

$$-\frac{\pi}{2} \leq \theta - \frac{3\pi}{8} \leq \frac{\pi}{2}$$

and

$$\sin\left(\theta - \frac{3\pi}{8}\right) = x$$

What we need to do is solve the inequalities

$$-\frac{\pi}{2} \leq \theta - \frac{3\pi}{8} \leq \frac{\pi}{2}$$

for θ.

Adding $3\pi/8$ throughout, we get

$$-\frac{\pi}{2} + \frac{3\pi}{8} \leq \theta \leq \frac{\pi}{2} + \frac{3\pi}{8}$$

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Now

$$-\frac{\pi}{2}+\frac{3\pi}{8}=\frac{4(-\pi)+1(3\pi)}{8}=\frac{-4\pi+3\pi}{8}=-\frac{\pi}{8}$$

and

$$\frac{\pi}{2}+\frac{3\pi}{8}=\frac{4(\pi)+1(3\pi)}{8}=\frac{4\pi+3\pi}{8}=\frac{7\pi}{8}$$

Therefore, for every x in the interval $[-1,1]$, there is a unique angle θ in the interval

$$\left[-\frac{\pi}{8},\frac{7\pi}{8}\right]$$

such that $\sin(\theta-(3\pi/8))=x$, so the range of an inverse of the function $\sin(\theta-(3\pi/8))$ is the interval $[-\pi/8,7\pi/8]$.

We'll now show that none of the other answer choices is the correct answer to the question.

Since the sine function is periodic, and in view of the need for uniqueness of $f^{-1}(x)$ for any element x of the domain of an invertible function f, the "size" of the range of any inverse of the function $\sin(\theta-(3\pi/8))$ must be equal to the "size" of the set given in answer choice C, which is just the length of the interval $[-\pi/8,7\pi/8]$. That length is

$$\frac{7\pi}{8}-\left(-\frac{\pi}{8}\right)=\frac{7\pi}{8}+\frac{\pi}{8}=\frac{8\pi}{8}=\pi$$

This is not the "size" of the set given in any of the other answer choices.

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The "size" of the set given in answer choice A is equal to the length of the interval $[-5\pi/8, -\pi/4]$, which is

$$-\frac{\pi}{4}-\left(-\frac{5\pi}{8}\right)=-\frac{\pi}{4}+\frac{5\pi}{8}=\frac{2(-\pi)+1(5\pi)}{8}=\frac{-2\pi+5\pi}{8}=\frac{3\pi}{8}\neq\pi$$

The set given in answer choice B consists of a pair of non-intersecting intervals, $[-\pi, -\pi/8]$ and $[0,7\pi/8]$, so the "size" of the set given in answer choice B is equal to the sum of the lengths of those two intervals. The length of the first interval is

$$-\frac{\pi}{8}-(-\pi)=-\frac{\pi}{8}+\pi=\frac{1(-\pi)+8(\pi)}{8}=\frac{-\pi+8\pi}{8}=\frac{7\pi}{8}$$

and the length of the second interval is clearly $7\pi/8$, so the "size" of the set in answer choice B is

$$\frac{7\pi}{8}+\frac{7\pi}{8}=\frac{14\pi}{8}=\frac{7\pi}{4}\neq\pi$$

The set given in answer choice D also consists of a pair of non-intersecting intervals, $[-\pi, -3\pi/8]$ and $[0,5\pi/8]$, so the "size" of the set given in answer choice D is equal to the sum of the lengths of those two intervals. The length of the first interval is

$$-\frac{3\pi}{8}-(-\pi)=-\frac{3\pi}{8}+\pi=\frac{1(-3\pi)+8(\pi)}{8}=\frac{-3\pi+8\pi}{8}=\frac{5\pi}{8}$$

and the length of the second interval is clearly $5\pi/8$, so the "size" of the set given in answer choice D is

$$\frac{5\pi}{8}+\frac{5\pi}{8}=\frac{10\pi}{8}=\frac{5\pi}{4}\neq\pi$$

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